Topic 5 –
Exact Equations

Suppose you have a first-order equation of the form

$$M(x,y) + N(x,y) \cdot y' = 0$$

And further suppose there exists a function $f(x,y)$ where

$$\frac{\partial f}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y)$$

Then we have that

$$M(x,y) + N(x,y) \cdot y' = 0$$

becomes

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

which is equivalent to

$$\frac{df}{dx} = 0$$

Math 2130

$f(x,y) \leftarrow$ f is function of x & y

$y = y(x) \leftarrow$ y is a function of x

chain rule:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \cdot \frac{d}{dx}(x) + \frac{\partial f}{\partial y} + \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

So for example the family of curves

$f(x,y)=c$ where c is a constant will then satisfy $\frac{\partial f}{\partial x}=0$ and hence $f(x,y)=c$ will give an implicit solution to the ODE.

Summary

If $\frac{\partial f}{\partial x}=M(x,y)$ and $\frac{\partial f}{\partial y}=N(x,y)$ then the family of curves

$$f(x,y)=c$$

where c is any constant will give implicit solutions to

$$M(x,y)+N(x,y)\cdot y'=0$$

When the above conditions are satisfied then we call $M(x,y)+N(x,y)y'=0$ an <u>exact equation</u>.

Ex: Consider the equation

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)} y' = 0$$

Let $f(x,y) = x^2y - y$.

We will see how to find this later

Then,

$$\frac{\partial f}{\partial x} = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

Thus, the equation

$$x^2y - y = c$$

gives an implicit solution to

$$2xy + (x^2-1)y' = 0.$$

In this case we can actually solve for y in terms of x and we get

We get $y = \dfrac{c}{x^2-1}$

Let's verify that this works.

We have

$$y = \frac{c}{x^2-1} = c\,(x^2-1)^{-1}$$

$$y' = -c\,(x^2-1)^{-2} \cdot (2x) = \frac{-2xc}{(x^2-1)^2}$$

Plugging these into

$$2xy + (x^2-1)\,y' = 0$$

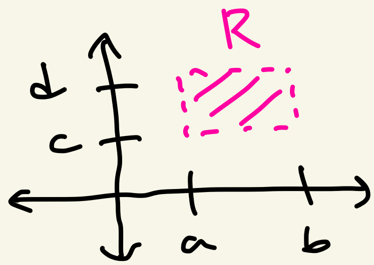
we get

$$2x\left(\frac{c}{x^2-1}\right) + (x^2-1)\left(\frac{-2xc}{(x^2-1)^2}\right) = 0$$

So we did indeed find a solution.

How do we know if we have an exact equation?

<u>Theorem</u>: Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in some rectangle R defined by $a<x<b$ and $c<y<d$



Then
$$M(x,y)+N(x,y)\cdot y'=0$$
is exact if and only if
$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Ex: With the previous equation

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)}y' = 0$$

we have that M and N are continuous everywhere and

$$\left.\begin{array}{ll} \dfrac{\partial M}{\partial x} = 2y & \dfrac{\partial M}{\partial y} = 2x \\[2em] \dfrac{\partial N}{\partial x} = 2x & \dfrac{\partial N}{\partial y} = 0 \end{array}\right\} \text{exist and are continuous everywhere}$$

Note that

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

So we know that

$$2xy + (x^2-1)y' = 0$$

is exact.

Now let's see how I found the f above.

We will give two methods.

$\boxed{\text{Method 1:}}$

We need $f(x,y)$ where

$$\boxed{\begin{aligned}\frac{\partial f}{\partial x} &= 2xy \quad (1)\\ \frac{\partial f}{\partial y} &= x^2-1 \quad (2)\end{aligned}} \longleftarrow \boxed{\begin{aligned}\frac{\partial f}{\partial x} &= M(x,y)\\ \frac{\partial f}{\partial y} &= N(x,y)\end{aligned}}$$

Integrate (1) with respect to x to get

$$f(x,y) = x^2y + \underbrace{C(y)}_{\text{constant with respect to } x} \quad (3)$$

Integrate (2) with respect to y to get

$$f(x,y) = x^2y - y + \underbrace{D(x)}_{\text{constant with respect to } y} \quad (4)$$

Now set (3) equal to (4) to get

$$\cancel{x^2y} + C(y) = \cancel{x^2y} - y + D(x)$$

So, $\underbrace{C(y)} = \underbrace{-y} + \underbrace{D(x)}_{0}$

Set $C(y) = -y$ and $D(x) = 0$.

Plug either of these into (3) or (4) to get f.

Say plug $C(y) = -y$ into (3) to get

$$f(x,y) = x^2y + C(y) = x^2y - y$$

So we get $f(x,y) = x^2y - y$ as before.

$\boxed{\text{Method 2}}$:

We want to solve

$$\boxed{\begin{aligned}\frac{\partial f}{\partial x} &= 2xy \quad (1)\\ \frac{\partial f}{\partial y} &= x^2-1 \quad (2)\end{aligned}}$$

Let's use equation (1) first.

Integrate

$$\frac{\partial f}{\partial x} = 2xy$$

with respect to x to get

$$f(x,y) = x^2y + g(y)$$

g is constant w/ respect to x

Then, differentiate with respect to y to get

$$\frac{\partial f}{\partial y} = x^2 + g'(y)$$

Thus, by equation (2) we get

$$x^2-1 = x^2 + g'(y).$$

So, $g'(y) = -1$.

Thus, $g(y) = -y$

you dont need a constant of integration here because we will set f to be equal to a constant

Therefore,

$$f(x,y) = x^2y + g(y) = x^2y - y$$

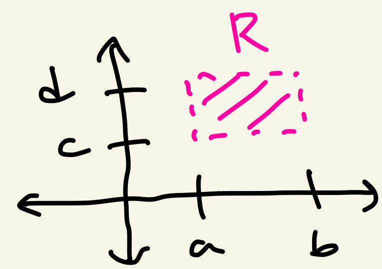
So,

$$f(x,y) = x^2y - y \quad \text{as before also.}$$

Below I put a proof of the main theorem in this topic. It's mainly for me But if you're interested, see below.

Let's prove this theorem.

Theorem: Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in some rectangle R defined by $a<x<b$ and $c<y<d$



Then
$$M(x,y)+N(x,y)\cdot y'=0$$
is exact if and only if
$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

proof: For simplicity suppose R is the entire xy-plane and that M and N are continuous for all (x,y) and so are their partial derivatives.

($\Rightarrow$) First suppose that $M+Ny'=0$ is exact. Then there exists f where $\frac{\partial f}{\partial x}=M$ and $\frac{\partial f}{\partial y}=N$.

Then, $\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f\right)=\frac{\partial N}{\partial x}$.

Calc III – Clairauts thm applied to M_y and N_x

($\Leftarrow$) Suppose now that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. We will show that this implies that $M + Ny' = 0$ is exact.

Since M is continuous we can define

$$f(x,y) = \int M(x,y)\,dx + g(y) \qquad (*)$$

where g is any function of y.

Here we get that $\boxed{\frac{\partial f}{\partial x} = M.}$

We want to now find $g(y)$ where

$$N = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M(x,y)\,dx + g'(y)$$

We will need

$$g'(y) = N - \frac{\partial}{\partial y} \int M(x,y)\,dx$$

To do this we can show that the RHS is just a function of y and hence we can integrate it with respect to y to get $g(y)$.

We have that

$$\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int M(x,y)\,dx\right) =$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int M(x,y)\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M(x,y)\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y} M$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

since $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$$= 0$$

Thus, such a $g(y)$ exists.

And

$$f(x,y) = \int M(x,y)\,dx + \int \left(N(x,y) - \int M(x,y)\,dx \right) dy$$

will satisfy $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$.

$\blacksquare$